A Control Theorist's Perspective on Dynamic Competitive Equilibria in Electricity Markets

Gui Wang, Anupama Kowli, Matias Negrete-Pincetic, Ehsan Shafieepoorfard, Sean Meyn *

* *Authors are with CSL and Dept. ECE, University of Illinois, Urbana-Champaign, IL 61820 USA (e-mail: meyn@illinois.edu).*

Abstract: We are moving towards a radical transformation of our energy systems. The success of the new paradigm created by the Smart Grid vision will require not only the creation and integration of new technologies into the grid, but also the redesign of the market structures coupled with it. In order to design the market structures for the grid of the future, economic models able to capture the new physical reality are the first requirement. In this paper, we present a general economic equilibrium model that refines standard economic models by including dynamics, uncertainty in supply and demand, and transmission constraints. The main finding is that the dynamical characteristics of the efficient equilibria can be highly undesirable for consumers, suppliers, or both. Moreover, we show that transmission constraints can exacerbate these characteristics.

Keywords: Economic equilibrium theory, electricity pricing, power transmission network.

1. INTRODUCTION

Power systems have always been a rich source of problems for control theorists. One of the most famous examples is the control of "singular perturbed dynamical systems", which was motivated in part by problems in power systems in the 1970s. Today there is an urgent need for models and control techniques to address a massively complex and hitherto unseen dynamical system consisting of new energy sources, information technologies and multiple market mechanisms (Massoud Amin and Wollenberg, 2005). We view control and system theory as one of the disciplines that will lead the research along this direction and provide valuable insights to comprehend and control this complex network, justifying the term "Smart Grid" (Santacana et al., 2010).

The focus of this paper is the market side of the Smart Grid. Our goal is to understand the impact of dynamics, constraints, and uncertainty, that will become more acute with the increased deployment of renewable resources. Such sentiment is aptly conveyed by Smith et. al. in the recent article (Smith et al., 2010), where the authors write that, "*little consideration was given to market design and operation under conditions of high penetrations of remote, variable renewable generation, such as wind ... and solar energy, which had not yet appeared on the scene in any significant amounts.*" Our approach is to characterize the competitive equilibria for a power network model that captures these complexities. This may be regarded as a stepping stone towards designing markets for the Smart Grid.

In the spirit of the cold war Milton Friedman wrote, *"Fundamentally, there are only two ways of coordinating the economic activities of millions. One is central direction involving the use of coercion – the technique of the army and of the modern totalitarian state. The other is voluntary cooperation of individuals – the technique of the marketplace"* (Friedman, 1962). We hope that most economists today would see this as an extreme point of view, but we find that polarization inhibits discussion on market design even today. Of course there are more than two ways! What is missing is a firm science for making design choices in a dynamic market, taking into account the special features of that particular market.

What are the special features of an electricity market? The electricity market is a coupling of two constrained and highly complex dynamical systems; one physical and one economic. The physical system is a complex network consisting of power flowing through transmission lines, modulated by distributed generation units, Kirchhoff's laws, and operational and security constraints. Loads and generation are each subject to uncertainty. The economic system typically consists of coupled markets such as day-ahead and real-time auctions. These auctions are dynamic — prices vary by two orders of magnitude in many real-time markets today — and are subject to uncertainty because of the "rational agents" driving the economic side. The presence of such complicating factors in these coupled systems, along with the sometimes orthogonal nature of the physical objectives with respect to the economic goals of the market players, make electricity market design a challenging task.

Control Theorist's Perspective? In a typical control design, an engineer starts with a simple model of the physical system, then chooses a control approach (e.g., PID, H_∞, LQG, MPC, adaptive, or MDP), and creates a feedback control solution. The design is usually verified via simulation and/or experiments. Even then, the design may need to be refined due to unexpected results in practice. If all efforts fail, then the engineer might consider a redesign of the physical system. A common feature in the design of electricity markets across several jurisdictions in the world has been an almost null consideration of any of these design steps. Typical market designs are guided by idealized models of the behavior of players and the underlying physical reality. However, it is clear now that the underlying physical reality of electricity generation can impact deeply the outcome of a given market structure.

In this paper, we present a general economic equilibrium model that refines standard economic models (Aliprantis et al., 2002) by including dynamics, uncertainty in supply and demand, and operational constraints associated with generation and transmission. The approach follows our earlier work (Cho and Meyn, 2010; Meyn et al., 2010) in which dynamic competitive equilibria are constructed under the most ideal assumptions in which "market power" is ruled out. This previous work is focused on a two-player market, without transmission constraints, and subject to special statistical assumptions. Using a Lagrangian decomposition that is standard in static economic analysis and certain dynamic economic analyses (Mas-Colell et al., 1995; Chow, 1997), we extend our prior work to address the economic impact of transmission constraints and other special features of the power grid, while maintaining the qualitative conclusions of our prior work. We find that the unique *efficient equilibrium* – the absolute optimum in an economic sense – may in fact be volatile and highly undesirable from the point of view of consumers, suppliers, or both. It may not be surprising that consumers will sometimes suffer from volatile prices. Less obvious is that suppliers may suffer from volatility. Consequently, they may not wish to stay in the market, which will difficult the achievement of resource adequacy – a component of reliability. To create a more reliable system we then must recognize that the classical notion of efficiency is just one metric. In the case of electricity markets, reliability is also critical for the economic security of the region.

While this paper does not provide a direct solution to the market design problem, it provides a framework for constructing dynamic models for electricity markets, and methods for characterizing the resulting competitive equilibria. The dynamic model is constructed using techniques well known in the control community and effectively captures the underlying physics of the power system while taking into considering the economic aspects of electricity trading.

A key finding is that standard economic conclusions hold only on average. We find that the dynamic behavior of prices can be highly volatile, especially when subject to transmission constraints. Our results highlight the need for moving beyond the widely adopted static economic models which typically provide insights only in terms of average quantities. An investigation beyond these average quantities is critical to design appropriate markets that can help to achieve economic and reliable power systems for sustainable development. We hope that the ideas and results presented in this work, complemented with recent research on economic theory of networks (Goyal, 2007), stochastic networks (Chen et al., 2006; Meyn, 2007), parameterized supply functions (Johari and Tsitsiklis, 2008), market dynamics (Kizilkale and Mannor, 2010; Roozbehani et al., 2010; Chen et al., 2010), load management schemes (Caramanis and Foster, 2009), and integration of renewables (Meyn et al., 2010) can facilitate the design and implementation of futures electricity markets.

The remainder of this paper contains four additional sections and is organized as follows. In Sec. 2 we present the economic and physical models of the electricity market. We devote Sec. 3 to characterize the competitive equilibrium in dynamic markets using a control-theorist-oriented theoretical scaffolding. The main results are the conditions for the existence of the competitive equilibrium in terms of duality concepts from nonlinear optimization theory. Characterization of the competitive equilibrium for several cases are presented in Sec. 4. We provide concluding remarks and final thoughts in Sec. 5.

2. ELECTRICITY MARKET MODEL

Restructuring has provided increased opportunities for competition in the electricity industry. In the market environment, the market participants make decisions based on their own interests. These self-interested entities compete in the markets for the rights to serve the load or be served. The electricity prices and the quantities sold are determined by competition in this market and the market rules.

Although the main reason for adopting energy markets has been to reduce electricity bills, reliability of service contines to remain an overriding concern of the system operators. As mentioned in the introduction, the reliability-driven operations – which are impacted by the physical constraints on the generation and the transmission – may conflict with the economic objectives of the market participants. Such conflicts are only intensified in a large power system consisting of multiple generators subject to minimum up/down-time constraints, ramping constraints and capacity constraints, connected to consumers via a complex, capacity-constrained transmission network. Since the aforementioned operational constraints on generation and transmission impact market decisions, we consider them explicitly in our analysis.

In what follows, we present a market model for energy in the perfect-competition setting of equilibrium economics. A critical assumption of this theory is that all players are 'price takers'. That is, no player can influence prices unilaterally. In every sense, this model is an appropriate representation of a perfect 'free-market' as analyzed in typical economics text. Not surprisingly, we find that market outcomes reflect the standard economic conclusions for efficient markets in which prices equal marginal costs, but only *on average*. Because of dynamic constraints, the sample path property of prices in the equilibrium will show perverse volatility patterns that have negative impact on consumers, suppliers or both.

2.1 The players and the rules

For simplicity we restrict discussion to a market consisting of a single "consumer" and a single "supplier" that represent price taking consumers and suppliers distributed across the grid. The model captures a power system consisting of N buses, indexed by $1, 2, ..., N$. For time $t \geq 0$, at each bus $n \in \{1, 2, ..., N\}$, there is an associated price $P_n(t)$ for energy traded at that bus. The *price-taking assumption* means that the price process $P_n(t)$ at a particular bus cannot be influenced by the actions of the consumers or suppliers.

Consumer We denote by $D_n(t)$ the demand at time t at bus n, and by $E_{\mathrm{D}n}(t)$ the energy withdrawn by the consumer at that bus. We assume that there is no free disposal for energy, which requires that $E_{\mathrm{D}n}(t) \leq D_n(t)$ for all t. If sufficient generation is available at bus n at time t, then $E_{\mathrm{D}n}(t) = D_n(t)$. In the event of insufficient generation, we have $E_{\mathrm{D}n}(t) < D_n(t)$, i.e., the consumer experiences a blackout.

The consumer obtains value on consuming energy and disutility for not meeting demand during a blackout. These are represented by possibly nonlinear functions,

$$\text{Utility of consumption:} \qquad v_n(E_{\mathrm{D}n}(t)), \tag{1a}$$
$$\text{Disutility of blackout:} \quad c_n^{\mathrm{bo}}(D_n(t) - E_{\mathrm{D}n}(t)) \tag{1b}$$

The consumer must pay for energy at price $P_n(t)$. We use $D(t)$, $E_{\mathrm{D}}(t)$, and $P(t)$ to denote the associated N-dimensional column vectors, and we use bold face font to denote the entire sample path. For instance, $\boldsymbol{P} := \{P(t) : t \geq 0\}$.

The welfare of the consumer at time t is the signed sum of his benefits and costs:

$$\mathcal{W}_{\mathrm{D}}(t) := \sum_n \big[v_n(E_{\mathrm{D}n}(t)) \; - \; c_n^{\mathrm{bo}}(D_n(t) - E_{\mathrm{D}n}(t)) \; - \; P_n(t)E_{\mathrm{D}n}(t)\big] \tag{2}$$

Observe that prices are determined by location. In the language of today's markets they are *locational prices.*

Supplier We denote by $E_{\mathrm{S}n}(t)$ and $R_{\mathrm{S}n}(t)$ the energy and reserve produced by the supplier at bus n. The operational and physical constraints on the production of energy and reserve are expressed abstractly as

$$(\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}) \in \mathsf{X}_{\mathrm{S}} \tag{3}$$

These constraints include ramping constraints on generation imposed by the physics of both generators and the grid. However, at this level of generality the specific details of the constraints are unimportant.

The production cost at time t for energy injected at bus n is denoted $c_n^{\mathrm{E}}(E_{\mathrm{S}n}(t))$, and for the reserve provided at that bus is $c_n^{\mathrm{R}}(R_{\mathrm{S}n}(t))$. The supplier receives the revenue $P_n E_{\mathrm{S}n}(t)$ for producing energy. The welfare of the supplier at time t is the difference between his revenue and the costs,

$$\mathcal{W}_{\mathrm{S}}(t) := \sum_n \big[P_n E_{\mathrm{S}n}(t) - c_n^{\mathrm{E}}(E_{\mathrm{S}n}(t)) - c_n^{\mathrm{R}}(R_{\mathrm{S}n}(t))\big] \tag{4}$$

Network To capture the impact of network constraints and exploit network structure we introduce a third player – the network. This is motivated in part by current practice: The transmission grid is operated by a third entity (neither the consumers nor the suppliers) in every electricity market operating in the world today.

We will find it convenient to introduce a "network welfare function" to define a competitive equilibrium for the power grid market model. The welfare of the network at time t represents the 'toll charges' for the transmission of energy. At time t, this is defined by,

$$\mathcal{W}_{\mathrm{T}}(t) := \sum_n \big[P_n(E_{\mathrm{D}n}(t) - E_{\mathrm{S}n}(t))\big] \tag{5}$$

The first constraint faced by the network is based on the assumption that it is lossless, so it neither generates nor consumes energy. Consequently, the network is subject to the supply-demand balance constraint,

$$1^T E_{\mathrm{S}}(t) = 1^T E_{\mathrm{D}}(t), \quad t \geq 0 \tag{6}$$

It is assumed that the buses are the nodes in a network and the links represent the transmission lines. There are L transmission lines, indexed by $\{1, 2, ..., L\}$. The network is assumed to be connected. We adopt a lossless DC model to represent the network. Suppose bus 1 is selected as the reference bus, based on which the *injection shift factor matrix* $H \in [-1, 1]^{N \times L}$ is defined, where H_{nl} denote the power distributed on line l when 1 MW is injected into bus n and withdrawn at the reference bus (Wood and Wollenberg, 1996; Chen et al., 2006).

Let $f_l^{\max}$ denote the capacity of transmission line l. On letting $H_l \in \mathbb{R}^N$ denote the l-th column of H, the capacity constraint for line l is expressed,

$$-f_l^{\max} \leq (E_{\mathrm{S}} - E_{\mathrm{D}})^T H_l \leq f_l^{\max} \tag{7}$$

2.2 Information and Uncertainty

In addition to the physical constraints captured by X_{S}, in every market there are informational constraints. In this paper, we adopt the highly idealized assumption that both sides of the market share a common information set. To model this, and also the impact of uncertainty and volatility, we opt for a stochastic model. Hence the processes described in the previous pages are all stochastic, and assumed to be adapted to a filtration $\{\mathcal{H}_t : t \geq 0\}$.

The consumer and supplier's objective function is the long-run discounted expected profit with discount rate γ, represented by

$$K_{\mathrm{D}} := \mathsf{E}\Big[\int e^{-\gamma t} \mathcal{W}_{\mathrm{D}}(t)\, dt\Big],$$
$$\text{and} \quad K_{\mathrm{S}} := \mathsf{E}\Big[\int e^{-\gamma t} \mathcal{W}_{\mathrm{S}}(t)\, dt\Big]$$

The supplier and consumer each aim to optimize their respective mean discounted mean welfare K_{S}, K_{D}. A similar expression can be obtained for K_{T}, the long-run discounted welfare of the network. These quantities will in general depend on the initial condition of the system. We suppress this dependency whenever possible.

To emphasize the similarity between static and dynamic equilibrium theory we adopt the following Hilbert-space notation: For two stochastic processes $\boldsymbol{F}$ and $\boldsymbol{G}$, each adapted to $\mathcal{H}_t$, we denote

$$\langle \boldsymbol{F}, \boldsymbol{G} \rangle := \mathsf{E}\Big[\int e^{-\gamma t} F(t) G(t)\, dt\Big]. \tag{8}$$

In particular, using this notation we have,

$$K_{\mathrm{S}} = \langle \boldsymbol{\mathcal{W}}_{\mathrm{S}}, \mathbf{1} \rangle \qquad \text{and} \qquad K_{\mathrm{D}} = \langle \boldsymbol{\mathcal{W}}_{\mathrm{D}}, \mathbf{1} \rangle \tag{9}$$

where **1** denotes the process that is identically unity.

Examples We first explain how this model is related to the single bus / single consumer model of (Cho and Meyn, 2010). In this model the decision variables were taken to be generation capacity $\boldsymbol{G}$ and reserve, where $\boldsymbol{G}$ coincides with $\boldsymbol{E}+\boldsymbol{R}$ in the notation of this paper. However, the generation was assumed normalized (the deviation from the day ahead market), so that negative values for $G(t)$ were possible. The set $\mathbf{X}^\circ_{\mathrm{S}}$ in (Cho and Meyn, 2010) would be defined by ramp constraints on $\boldsymbol{G}$: For all $t_1 > t_0 \geq 0$,

$$\zeta^- \leq \frac{E_{\mathrm{S}}(t_1) - E_{\mathrm{S}}(t_0)}{t_1 - t_0} + \frac{R_{\mathrm{S}}(t_1) - R_{\mathrm{S}}(t_0)}{t_1 - t_0} \leq \zeta^+ . \qquad (10)$$

In fact, in the model of (Cho and Meyn, 2010) the lower bound was relaxed, so that $\zeta^- = \infty$.

In this prior work the utility of consumption was assumed to be of the form $v \min(G, D)$ for a constant $v > 0$. The disutility of blackout was taken to be piecewise linear: zero for $R > 0$, and proportional to $D - G$ otherwise. The analogous functions for the model introduced here are the linear and piecewise linear functions,

$$\begin{aligned} \textit{Value of consumption} &= vE \\ \textit{Cost of blackout} &= c^{\mathrm{bo}} \max(D - E, 0)\,, \end{aligned} \qquad (11)$$

where v and c^{bo} are constants.

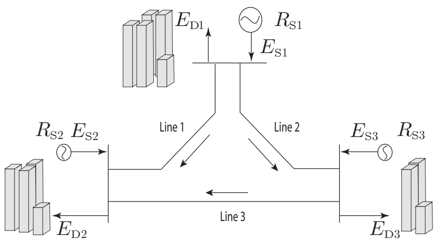

Fig. 1. Three-node power transmission network.

Fig. 1 shows a three-bus model considered in several other market analysis papers (Chen et al., 2006). If the impedances are identical in the three transmission lines, and bus 1 is chosen as the reference bus, then the injection shift factor matrix is given by

$$H = \tfrac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & -1 \\ -1 & -1 & -2 \end{bmatrix} \qquad (12)$$

We return to this example in Sec. 4.2.

3. EQUILIBRIUM AND EFFICIENCY IN DYNAMIC MARKETS

The competitive equilibrium of economics is used as a vehicle to study the outcomes of a market under a set of idealized assumptions (Aliprantis et al., 2002). It is a widely accepted benchmark for evaluating real markets outcomes. If the behavior of a market nearly matches the behavior predicted by the competitive equilibrium, then the market is deemed to be functioning well. The existence and optimality of a competitive equilibrium have been studied extensively using static models, under standard assumptions, typically including continuity and convexity of cost and utility function.

In the electricity industry, physical and operational limits of the facilities and network impose stringent constraints on the behavior of the market participants. This can present challenges when applying the usual market analysis tools to evaluate the market. A complete understanding of the impact of these physical characteristics on market outcomes remains an open question. We believe that a better science for dynamic, networked markets is a key requirement for the electricity market designs of the future.

The usual definition of a competitive equilibrium with two players is based on the respective optimization problems of the supplier and consumer,

$$(\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}) \in \mathop{\arg\max}_{\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}} \langle \boldsymbol{\mathcal{W}}_{\mathrm{S}}, \mathbf{1} \rangle, \qquad (13)$$

$$\boldsymbol{E}_{\mathrm{D}} \in \mathop{\arg\max}_{\boldsymbol{E}_{\mathrm{D}}} \langle \boldsymbol{\mathcal{W}}_{\mathrm{D}}, \mathbf{1} \rangle\,, \qquad (14)$$

where the welfare functions are given in (4,2). We adopt the same conventions in our dynamic analysis. However, in the equilibrium definition that follows we introduce the third player – the network – to account for network constraints (and also rule out 'arbitrage' – an issue to be illustrated with examples in a lengthier version of this paper under preparation).

Definition 1. A *competitive equilibrium* is a quadruple of process vectors: consumed energy, supplied energy, supplied reserve, and energy price, denoted as $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$, which satisfies the following conditions:

(i) $(\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}})$ solves (13), subject to the operational/physical constraint (3).
(ii) $\boldsymbol{E}_{\mathrm{D}}$ solves (14).
(iii) The pair $(\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}})$ optimizes the welfare function of the network,

$$(\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}) \in \mathop{\arg\max}_{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}} \langle \boldsymbol{\mathcal{W}}_{\mathrm{T}}, \mathbf{1} \rangle, \qquad (15)$$

subject to the supply-demand balance constraint (6), and the network constraint (7).

The supplier, consumer and network are also subject to the measurability constraints outlined in Sec. 2.2 in their respective optimization problems. □

Note that the set of feasible strategies for the supplier is subject to the operational/physical constraints (3), but the consumer's optimization problem is *not* subject these constraints.

To evaluate the welfare performance of the market, we introduce into our analysis a *social planner* who aims to maximize the economic well-being of everyone in the system. The social planner uses the total welfare, denoted by $\mathcal{W}_{\mathrm{tot}}(t)$, to measure the economic well-being of the system with

$$\mathcal{W}_{\mathrm{tot}}(t) := \mathcal{W}_{\mathrm{S}}(t) + \mathcal{W}_{\mathrm{D}}(t) + \mathcal{W}_{\mathrm{T}}(t) \qquad (16)$$

Note that $\mathcal{W}_{\mathrm{tot}}(t)$ is the sum of $\{v_n(E_{\mathrm{D}n}(t)) - c^{\mathrm{bo}}_n(D_n(t) - E_{\mathrm{D}n}(t)) - c^{\mathrm{E}}_n(E_{\mathrm{S}n}(t)) - c^{\mathrm{R}}_n(R_{\mathrm{S}n}(t))\}$ over all nodes n.

Definition 2. The social planner's problem (SPP) is

$$\max_{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}} \langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1} \rangle, \qquad (17)$$

subject to the operational/physical constraint (3), the network constraint (7), and energy-balance constraint (6). Its solution is called an *efficient allocation.* □

We assume throughout the paper that the SPP (17) has a solution, denoted $(\boldsymbol{E}_{\mathrm{D}}^*, \boldsymbol{E}_{\mathrm{S}}^*, \boldsymbol{R}_{\mathrm{S}}^*)$.

Let $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$ be a competitive equilibrium. If $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ is an efficient allocation, then we say that the equilibrium is efficient. If every competitive equilibrium is efficient, then we say that the *first welfare theorem* holds. On the other hand, let $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ be an efficient allocation. If we can construct a price process $\boldsymbol{P}$ such that $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$ becomes a competitive equilibrium, we say the allocation is supported by the price $\boldsymbol{P}$. If every efficient allocation can be supported, then we say that the *second welfare theorem* holds.

We now analyze the competitive equilibrium of the market in the context of the welfare theorems. We start the analysis by constructing the Lagrangian of the SPP.

Definition 3. The Lagrangian of the SPP is

$$\begin{aligned}\mathcal{L} = &- \langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1}\rangle + \langle \boldsymbol{\lambda}, (1\cdot \boldsymbol{E}_{\mathrm{D}} - 1\cdot \boldsymbol{E}_{\mathrm{S}})\rangle \\ &+ \sum_l \langle \boldsymbol{\mu}_l^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})\cdot H_l - f_l^{\max}\rangle \\ &+ \sum_l \langle \boldsymbol{\mu}_l^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})\cdot H_l - f_l^{\max}\rangle\end{aligned}$$

where $\mu_l^+(t) \geq 0$ and $\mu_l^-(t) \geq 0$ for all t and l. □

A key step is to define the candidate price process $\boldsymbol{P}$ as

$$P_n(t) := \lambda(t) + \sum_l (\mu_l^-(t) - \mu_l^+(t)) H_{ln}, \quad t \geq 0,\ n \geq 1. \tag{18}$$

From the definitions, we conclude that the Lagrangian can be expressed,

$$\begin{aligned}\mathcal{L} = &- \sum_n \{\langle v_n(\boldsymbol{E}_{\mathrm{D}n}) - c_n^{\mathrm{bo}}(\boldsymbol{D}_n - \boldsymbol{E}_{\mathrm{D}n}), \mathbf{1}\rangle - \langle \boldsymbol{P}_n, \boldsymbol{E}_{\mathrm{D}n}\rangle\} \\ &- \sum_n \{\langle \boldsymbol{P}_n, \boldsymbol{E}_{\mathrm{S}n}\rangle - \langle c_n^{\mathrm{E}}(\boldsymbol{E}_{\mathrm{S}n}) + c_n^{\mathrm{R}}(\boldsymbol{R}_{\mathrm{S}n}), 1\rangle\} \\ &- \sum_l \langle \boldsymbol{\mu}_l^+ + \boldsymbol{\mu}_l^-, f_l^{\max}\rangle\end{aligned} \tag{19}$$

Therefore, $\mathcal{L}$ is a constant minus the sum of supplier and consumer welfare functions, with $\mathcal{W}_{\mathrm{D}}$ and $\mathcal{W}_{\mathrm{D}}$ defined using this price $\boldsymbol{P}$.

Definition 4. The dual functional for the SPP is

$$h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) = \min_{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}} \mathcal{L} \tag{20}$$

The following weak duality bound follows since the minimization in (20) amounts to a relaxation of the SPP (17):

Lemma 5. (Weak Duality). For any allocation $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ and Lagrangian multiplier $(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$ with $\boldsymbol{\mu}^+, \boldsymbol{\mu}^- \geq 0$, we have

$$-\langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1}\rangle \geq h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) \tag{21}$$

An equality in (21) implies that strong duality holds. The main result of this section characterizes the existence of a competitive equilibrium in terms of strong duality:

Theorem 6. (Existence of Competitive Equilibrium). The market admits a competitive equilibrium if and only if the SPP satisfies strong duality.

Proof. We first prove the *sufficient condition*: strong duality implies existence of competitive equilibrium. Since strong duality holds, we have

$$-\langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1}\rangle = h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) \tag{22}$$

Suppose that the allocation $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ is feasible for the SPP. We then construct a competitive equilibrium with price as given in (18).

The feasibility of the triple $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ for SPP implies $1^T E_{\mathrm{S}}(t) = 1^T E_{\mathrm{D}}(t)$ for all t, and hence

$$\begin{aligned}\mathcal{L} = -\langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1}\rangle &+ \sum_l \langle \boldsymbol{\mu}_l^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})\cdot H_l - f_l^{\max}\rangle \\ &+ \sum_l \langle \boldsymbol{\mu}_l^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max}\rangle\end{aligned}$$

Feasibility also implies $-f_l^{\max} \leq (E_{\mathrm{S}}(t) - E_{\mathrm{D}}(t))^T H_l \leq f_l^{\max}$, and given the non-negativity of $\boldsymbol{\mu}^+, \boldsymbol{\mu}^-$, we have

$$\begin{aligned}&\langle \boldsymbol{\mu}^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max}\rangle \leq 0\,, \\ \text{and } &\langle \boldsymbol{\mu}^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max}\rangle \leq 0\end{aligned}$$

This together with (22) gives $\mathcal{L} \leq h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$ But, by the definition in (20), we have $\mathcal{L} \geq h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$, so that we obtain the identity,

$$h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) = \mathcal{L} \tag{23}$$

This identity implies that $\boldsymbol{E}_{\mathrm{D}}$ maximizes the consumer's welfare, $\{\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ maximizes the supplier's welfare, and

$$\begin{aligned}&\langle \boldsymbol{\mu}^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max}\rangle = 0\,, \\ \text{and } &\langle \boldsymbol{\mu}^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max}\rangle = 0\end{aligned}$$

Using the prices $\{\boldsymbol{P}_n = \boldsymbol{\lambda} + \sum_l (\boldsymbol{\mu}_l^- - \boldsymbol{\mu}_l^+) H_{ln}\}$ defined in (18), we substitute the above two equations into the network welfare expression to obtain,

$$\begin{aligned}&\Big\langle \sum_n [\boldsymbol{P}_n(\boldsymbol{E}_{\mathrm{D}n} - \boldsymbol{E}_{\mathrm{S}n})], \mathbf{1}\Big\rangle \\ =&\Big\langle \sum_n \Big[\sum_l (\boldsymbol{\mu}_l^- - \boldsymbol{\mu}_l^+) H_{ln}(\boldsymbol{E}_{\mathrm{D}n} - \boldsymbol{E}_{\mathrm{S}n})\Big], \mathbf{1}\Big\rangle \\ =&\Big\langle \sum_l (\boldsymbol{\mu}_l^+ + \boldsymbol{\mu}_l^-)\cdot f_l^{\max}, \mathbf{1}\Big\rangle\,.\end{aligned}$$

This is independent of $\{\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$, which implies that the welfare of the network is maximized under the prices $\{\boldsymbol{P}_n\}$. Thus, we conclude that $\boldsymbol{P}$ as defined in (18) is the equilibrium price as claimed.

Next, we prove the *necessary condition*: existence of competitive equilibrium implies strong duality. Suppose that $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$ is a competitive equilibrium. Then we know that $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}\}$ maximizes the network welfare when the price is $\boldsymbol{P}$. The Lagrangian associated with the maximization of network welfare is expressed as follows: For any $\boldsymbol{\mu}^+, \boldsymbol{\mu}^- \geq 0$,

$$\begin{aligned}\mathcal{L}_{\mathrm{T}} = &- \sum_n \langle \boldsymbol{P}_n, (\boldsymbol{E}_{\mathrm{D}n} - \boldsymbol{E}_{\mathrm{S}n})\rangle + \langle \boldsymbol{\lambda}, (1^T \boldsymbol{E}_{\mathrm{D}} - 1^T \boldsymbol{E}_{\mathrm{S}})\rangle \\ &+ \sum_l \langle \boldsymbol{\mu}_l^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max}\rangle \\ &+ \sum_l \langle \boldsymbol{\mu}_l^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max}\rangle\end{aligned} \tag{24}$$

The maximization of network welfare is a linear program, and hence the optimum satisfies the KKT conditions. As a consequence, associated with the constraints $\dfrac{\partial \mathcal{L}_{\mathrm{T}}}{\partial E_{\mathrm{D}n}} = \dfrac{\partial \mathcal{L}_{\mathrm{T}}}{\partial E_{\mathrm{S}n}} = 0$, there exist $\{\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-\}$ such that (18) holds:

$$\boldsymbol{P}_n = \boldsymbol{\lambda} + \sum_l (\boldsymbol{\mu}_l^- - \boldsymbol{\mu}_l^+) H_{ln}$$

Moreover, by complementary-slackness, we have

$$\begin{aligned}\langle \boldsymbol{\mu}^+, (\boldsymbol{E}_\mathrm{S} - \boldsymbol{E}_\mathrm{D})^T H_l - f_l^{\max}\rangle = 0 \\ \langle \boldsymbol{\mu}^-, -(\boldsymbol{E}_\mathrm{S} - \boldsymbol{E}_\mathrm{D})^T H_l - f_l^{\max}\rangle = 0\end{aligned} \tag{25}$$

Next, we investigate $h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$ for the SPP, using the multipliers from the maximization of network welfare. Since $\{\boldsymbol{E}_\mathrm{D}, \boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}, \boldsymbol{P}\}$ is a competitive equilibrium, $\boldsymbol{E}_\mathrm{D}$ maximizes the consumer's welfare, and $\{\boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}\}$ maximizes the supplier's welfare. Based on the form (18) for $\boldsymbol{P}$ we conclude that

$$\{\boldsymbol{E}_\mathrm{D}, \boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}\} \in \mathop{\arg\min}_{\boldsymbol{E}_\mathrm{D}, \boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}} \mathcal{L}$$

Substituting $\{\boldsymbol{E}_\mathrm{D}, \boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}\}$ into the definition of $\mathcal{L}$, and applying the complementary slackness equation (25),

$$-\langle \mathcal{W}_\mathrm{tot}, \mathbf{1}\rangle = h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$$

That is, strong duality holds. □

We stress that the theorem characterizes prices in any competitive equilibrium:

Corollary 7. The only candidates for prices in a competitive equilibrium are given by (18), based on the optimal Lagrangian multipliers.

Proof. This is because only the optimal multipliers could possibly lead to strong duality. □

Theorem 6 tells us that computation of prices and quantities can be decoupled: The quantities $\{\boldsymbol{E}_\mathrm{D}, \boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}\}$ are obtained through the solution of the SPP, and the price process $\boldsymbol{P}^e$ is obtained as a solution to its dual. The following corollary underlines this point. If $\boldsymbol{P}^e$ supports one competitive equilibrium, then it supports any other competitive equilibrium.

Corollary 8. If $\{\boldsymbol{E}^1_\mathrm{D}, \boldsymbol{E}^1_\mathrm{S}, \boldsymbol{R}^1_\mathrm{S}, \boldsymbol{P}^1\}$ and $\{\boldsymbol{E}^2_\mathrm{D}, \boldsymbol{E}^2_\mathrm{S}, \boldsymbol{R}^2_\mathrm{S}, \boldsymbol{P}^2\}$ are two competitive equilibria, then $\{\boldsymbol{E}^2_\mathrm{D}, \boldsymbol{E}^2_\mathrm{S}, \boldsymbol{R}^2_\mathrm{S}, \boldsymbol{P}^1\}$ is also a competitive equilibrium.

Proof. Due to the necessary condition of Theorem 6 there exists Lagrange multipliers $(\boldsymbol{\lambda}_1, \boldsymbol{\mu}^+_1, \boldsymbol{\mu}^-_1)$ and $(\boldsymbol{\lambda}_2, \boldsymbol{\mu}^+_2, \boldsymbol{\mu}^-_2)$ corresponding to the two equilibria, such that

$$\begin{aligned}h(\boldsymbol{\lambda}_1, \boldsymbol{\mu}^+_1, \boldsymbol{\mu}^-_1) = -\langle \mathcal{W}_{\mathrm{tot}\mathbf{1}}, \mathbf{1}\rangle = \\ h(\boldsymbol{\lambda}_2, \boldsymbol{\mu}^+_2, \boldsymbol{\mu}^-_2) = -\langle \boldsymbol{\mathcal{W}}_{\mathrm{tot2}}, \mathbf{1}\rangle\end{aligned}$$

By the sufficient condition of Theorem 6 any of these price and quantity pair satisfying strong duality will constitute a competitive equilibrium. □

The first and second fundamental theorems of welfare economics are each implied by Theorem 6.

Theorem 9. (First Fundamental Theorem). Any competitive equilibrium, if it exists, is efficient.

Proof. By the proof of necessary condition, for any competitive equilibrium $\{\boldsymbol{E}_\mathrm{D}, \boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}, \boldsymbol{P}\}$, there exists some $(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$, such that

$$h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) = -\langle \mathcal{W}_\mathrm{tot}, \mathbf{1}\rangle$$

By weak duality (21), $\{\boldsymbol{E}_\mathrm{D}, \boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}\}$ is an efficient allocation. □

The second welfare follows similarly:

Theorem 10. (Second Fundamental Theorem). If the market admits a competitive equilibrium, then for any efficient allocation $\{\boldsymbol{E}_\mathrm{D}, \boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}\}$, there exists a supporting price process $\boldsymbol{P}$ such that $\{\boldsymbol{E}_\mathrm{D}, \boldsymbol{E}_\mathrm{S}, \boldsymbol{R}_\mathrm{S}, \boldsymbol{P}\}$ constitutes a competitive equilibrium. □

4. EQUILIBRIUM PRICES

In the model of (Cho and Meyn, 2010) we found that the price process $\boldsymbol{P}^e$ in the competitive equilibrium is given by,

$$P^e(t) = (v + c^\mathrm{bo})\mathbb{I}\{R^*(t) \ge 0\}, \tag{26}$$

where $\boldsymbol{R}^*$ is the reserve process in the solution to the SPP. The quantity $v + c^\mathrm{bo}$ is known as the *choke-up price* since it is the maximum the consumer is willing to pay. The choke-up price will be extremely large in any realistic power system, so that these prices show tremendous volatility. However, in this prior work it was shown that the *average* price coincides with marginal cost c for generation, in the sense that

$$\gamma \mathsf{E}\Big[\int e^{-\gamma t} P^e(t)\, dt\Big] = c \tag{27}$$

The expression (27) required that the initial reserves be sufficiently large. The derivation was by direct calculation, based on the assumption that the consumer is not subject to temporal constraints.

In this section we show that the same conclusions can be derived for the general model using Lagrange multiplier techniques. The conclusions will be slightly different because of one deviation from the model of (Cho and Meyn, 2010) : In this prior work the generation $G(t) := E(t) + R(t)$ was assumed to be constrained by ramp rate, but no other constraints were imposed. In particular, negative values were allowed since the generation $G(t)$ was assumed to be normalized (the deviation from the power allocation determined in the day-ahead market).

We begin with the single-producer single-consumer model without transmission constraints.

4.1 Equilibrium prices: single bus and single consumer

We first establish the formula for $\boldsymbol{P}^e$. We assume throughout that $v \colon \mathbb{R}_+ \to \mathbb{R}_+$ and $c^\mathrm{bo} \colon \mathbb{R} \to \mathbb{R}_+$ are continuously differentiable functions of their arguments.

Proposition 11. Suppose that $(\boldsymbol{E}^*, \boldsymbol{R}^*)$ is a solution to the SPP that defines a competitive equilibrium with price process $\boldsymbol{P}^e$. Then,

$$P^e(t) = \nabla v\,(E^*(t)) + \nabla c^\mathrm{bo}\,(D(t) - E^*(t)), \quad t \ge 0. \tag{28}$$

Proof. In the single bus model we have $\mathcal{W}_\mathrm{D}(t) := v(E_\mathrm{D}(t)) - c^\mathrm{bo}(D(t) - E_\mathrm{D}(t)) - P^e(t)E_\mathrm{D}(t)$. The formula follows because $\boldsymbol{E}^* = \boldsymbol{E}_\mathrm{D}$ in the competitive equilibrium, and the consumer is myopic (recall that the consumer does not consider ramp constraints). □

To obtain a formula for the average price we must consider the optimization problem posed by the supplier. For simplicity we assume that $\mathsf{X}^\circ_\mathrm{S}$ is defined as in (Cho and Meyn, 2010) by the ramp constraints (10), and subject to the non-negativity constraints $E_\mathrm{S}(t) \ge 0$, $R_\mathrm{S}(t) \ge 0$, for all t.

We then consider a Lagrangian relaxation, in which the constraint $E_\mathrm{S}(0) + R_\mathrm{S}(0) = g_0$ is captured in the Lagrange multiplier ν. For this we define the Lagrangian,

$$\mathcal{L}(\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S},\nu)= \quad \mathsf{E}\Big[\int e^{-\gamma t}\mathcal{W}_{\rm D}(t)\,dt\Big] \\ -\nu[E_{\rm S}(0)+R_{\rm S}(0)-g_0] \tag{29}$$

The following result is a consequence of the local Lagrange multiplier theorem (Luenberger, 1969).

Lemma 12. Suppose $(\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S})$ maximizes $\mathcal{L}(\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S},0)$ over pairs in $\mathsf{X}_{\rm S}^{g_0}$. Then, there exist $\nu^*\in\mathbb{R}$ such that $(\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S})$ maximizes $L_1(\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S},\nu^*)$ over the larger set of functions $\mathsf{X}_{\rm S}^{\circ}$. □

The next result is a construction required in an application of the Lagrange multiplier result Lemma 12.

Lemma 13. Suppose that $(\boldsymbol{R}_{\rm S},\boldsymbol{E}_{\rm S})$ belongs to $\mathsf{X}_{\rm S}^{\circ}$. Then there exists a family of solutions $\{(\boldsymbol{R}_{\rm S},\boldsymbol{E}_{\rm S}^{\alpha}) : |\alpha|\le 1\}\subset \mathsf{X}_{\rm S}^{\circ}$ satisfying $E_{\rm S}^{\alpha}(0)=\max(E_{\rm S}(0)+\alpha,0)$, $|E_{\rm S}^{\alpha}(t)-E_{\rm S}(t)|\le \alpha$ for all α, and for $t>0$,

$$\lim_{\alpha\to 0}\frac{1}{\alpha}(E_{\rm S}^{\alpha}(t)-E_{\rm S}(t))=\mathbf{1}_{\rm S}^{+}(t):=\mathbb{I}\{E_{\rm S}(t)>0\}\,.$$

Combining the two Lemmas we easily obtain the following extension of (27):

Theorem 14. Suppose that $(\boldsymbol{E}^*,\boldsymbol{R}^*)$ is a solution to the SPP that defines a competitive equilibrium with price process $\boldsymbol{P}^e$. Suppose that $E^*(0)>0$, and suppose moreover the following bounds hold: The processes $\boldsymbol{E}^*$ is square integrable, meaning that $\langle \boldsymbol{E}^*,\boldsymbol{E}^*\rangle<\infty$, and the cost function $c_{\rm S}$ satisfies $c_{\rm S}(e)+|\nabla c_{\rm S}(e)|\le c_0(1+e^2)$ for some $c_0>0$ and all $e\ge 0$.

Then the average price coincides with average marginal cost of energy plus the scaled sensitivity term ν^*:

$$\gamma\mathsf{E}\Big[\int_0^{\infty} e^{-\gamma t}\mathbf{1}_{\rm S}^{+}(t)P^e(t)\,dt\Big] \\ =\gamma\mathsf{E}\Big[\int_0^{\infty} e^{-\gamma t}\mathbf{1}_{\rm S}^{+}(t)\nabla c_{\rm S}\left(E^*(t)\right)dt\Big]+\gamma\nu^* \tag{30}$$

Proof. The Lagrangian $\mathcal{L}(\boldsymbol{E}^{\alpha},\boldsymbol{R}^*,\nu)$ is differentiable as a function of α under the assumptions of the theorem, and we have the expression,

$$\frac{d}{d\alpha}\mathcal{L}(\boldsymbol{E}^{\alpha},\boldsymbol{R}^*,\nu^*)= \quad \mathsf{E}\Big[\int e^{-\gamma t}\frac{d}{d\alpha}\mathcal{W}_{\rm S}^{\alpha}(t)\,dt\Big] \\ -\frac{d}{d\alpha}\nu^*[E^{\alpha}(0)+R^*(0)-g_0] \tag{31}$$

where $\mathcal{W}_{\rm S}^{\alpha}(t):=P^e(t)E_{\rm S}^{\alpha}(t)-c_{\rm S}^{\rm E}(E_{\rm S}^{\alpha}(t))-c_{\rm R}^{\rm E}(R^*(t))$. In this calculation the square integrability assumption and bounds on $c_{\rm S}$ are used to justify taking the derivative under the expectation and integral in (29).

The conclusion of the theorem then follows from two facts: First is optimality of the Lagrangian at $\alpha=0$, giving $\frac{d}{d\alpha}\mathcal{L}(\boldsymbol{E}^{\alpha},\boldsymbol{R}^*,\nu)=0$ for $\alpha=0$. We then apply Lemma 13 which allows an application of the chain rule to obtain,

$$\frac{d}{d\alpha}\mathcal{W}_{\rm S}^{\alpha}(t)\Big|_{\alpha=0}=\mathbf{1}_{\rm S}^{+}(t)\big(P^e(t)E^*(t)-\nabla c_{\rm S}\left(E^*(t)\right)\big)$$

Evaluating (31) at $\alpha=0$ then gives the desired result. □

The formula (30) is very similar to (27). The main difference is the modification when energy is zero, and the introduction of the term $\gamma\nu^*$. The latter is not present in (27) since this result required large initial reserves, which implies that $\nu^*=0$.

Once again we conclude that, although prices may show high volatility in the competitive equilibrium, under general conditions we find that the average price is precisely the average marginal cost. The price depends on the initial value g_0, and this dependence is captured through the sensitivity term ν^*.

4.2 Equilibrium prices: markets with network constraints

We now turn to the general electricity market model subject to network constraints. We assume this market has a competitive equilibrium denoted $\{\boldsymbol{E}_{\rm D},\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S},\boldsymbol{P}^e\}$; our goal is to identify properties of $\boldsymbol{P}^e$ as was accomplished for the scalar model in Sec. 4.1.

Our analysis is based on Lagrangian relaxations of the market model. For this reason we extend the domain of the utility and disutility functions $\{v_n(\,\cdot\,),c_n^{\rm bo}(\,\cdot\,)\}$ to all of $\mathbb{R}$, and we assume throughout that these functions are continuously differentiable.

The following fictitious market is key to our analysis.

Definition 15. The $\not\$$-market is defined as follows:

- (i) The consumer and transmission models are unchanged. The operational/physical constraints (3) on $(\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S})$ are relaxed.
- (ii) The welfare functions of the consumer and the network are unchanged.
- (iii) The welfare function of the supplier is *identically zero.* This is achieved by overriding the production cost functions as follows:

$$c_n^{\rm E\not\$}(E_{{\rm S}n}(t))=P_n^e(t)E_{{\rm S}n}(t),\quad c_n^{\rm R\not\$}(R_{{\rm S}n}(t))=0 \tag{32}$$

Since the welfare function $\mathcal{W}_{\rm S}$ for the supplier is identically zero in this model, the market essentially reduces to a model consisting of two players: the consumer and the network. We find that the equilibrium for the original market model provides an equilibrium for the two-agent market:

Lemma 16. $\{\boldsymbol{E}_{\rm D},\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S},\boldsymbol{P}^e\}$ is a competitive equilibrium for the $\not\$$-market.

Proof. The triple $\{\boldsymbol{E}_{\rm D},\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S}\}$ satisfies the supply-demand balance constraint, and the network constraints, and maximizes the consumer and the network welfare functions under price $\boldsymbol{P}^e$. To prove the Lemma it remains to show that $\{\boldsymbol{E}_{\rm S},\boldsymbol{R}_{\rm S}\}$ maximizes the supplier's welfare in the $\not\$$-market. This is immediate since the supplier's welfare function is independent of the decision of the supplier — it is always zero by assumption. □

Lemma 16 is the main motivation for the introduction of the $\not\$$-market. We have seen that the price $\boldsymbol{P}^e$ in the original market model supports a competitive equilibrium in the $\not\$$-market. Hence we can hope to extract properties of $\boldsymbol{P}^e$ in the simpler market model. The following result shows that the $\not\$$-market is indeed very simple. The result is a consequence of the assumption that there are no temporal constraints in this model.

Lemma 17. All players, as well as the social planner, are myopic in the $\not\$$-market. □

Recall that in the single bus model, the derivation of the supporting price (28) was based on the assumption that consumers are not subject to temporal constraints. Lemma 17 justifies the same approach to analysis in the network model. The optimization problems posed by the consumer and the network in the \$-market are reduced to a "snapshot model" in which we can fix a time t to obtain properties of $P^e(t)$, exactly as in the derivation of (28).

With t fixed, the snapshot optimization problem posed by the social planner problem (SPP) is defined by the maximum of the total welfare $\mathcal{W}^{\$}_{\text{tot}} =$

$$\sum_n \Big[v_n(E_{\text{D}n}) - c^{\text{bo}}_n(D_n - E_{\text{D}n}) - P^e_n E_{\text{S}n} \Big] \qquad (33)$$

subject to the following constraints:

$$\begin{aligned} 1^T E_S &= 1^T E_D && \leftrightarrow \lambda \\ -f^{\max}_l &\le (E_{\text{S}} - E_{\text{D}})^T H_l \le f^{\max}_l && \leftrightarrow \mu^-_l, \mu^+_l \ge 0 \\ 0 \le E_{\text{D}n} &\le D_n && \leftrightarrow \eta^-_n, \eta^+_n \ge 0 \end{aligned}$$

The terms on the right hand side are Lagrange multipliers corresponding to the given constraints. The Lagrangian of the SPP for the \$-market is the function of static variables:

$$\begin{aligned} \mathcal{L}^h = & -\mathcal{W}^{\$}_{\text{tot}} + \lambda(1^T(E_{\text{D}} - E_{\text{S}})) \\ & + \sum_l \mu^+_l [(E_{\text{S}} - E_{\text{D}})^T H_l - f^{\max}_l] \\ & + \sum_l \mu^-_l [-(E_{\text{S}} - E_{\text{D}})^T H_l - f^{\max}_l] \qquad (34) \\ & + \sum_n \eta^+_n (E_{\text{D}n} - D_n) - \sum_n \eta^+_n E_{\text{D}n} \end{aligned}$$

where $\mu^-_l, \mu^+_l, \eta^-_n, \eta^+_n \ge 0$.

Proposition 18. Consider the SPP for the \$-market with welfare function defined in (33). Suppose that μ^-_l, μ^+_l, η^-_n, η^+_n are the non-negative, optimal solutions to the dual with Lagrangian (34). Then the equilibrium price has entries given as follows: For $n = 1, \ldots, N$,

$$P^e_n = \nabla v_n(E_{\text{D}n}) + \nabla c^{\text{bo}}_n(D_n - E_{\text{D}n}) + \Lambda, \qquad (35)$$

where $P^e_n = P^e_n(t)$, $E_{\text{D}n}, D_n, E_{\text{D}n}$ are also the variables observed at time t, and where

$$\Lambda = \begin{cases} 0, & 0 < E_{\text{D}n} < D_n \\ -\eta^+_n, & E_{\text{D}n} = D_n \\ \eta^-_n & E_{\text{D}n} = 0 \end{cases}$$

Proof. Since $\{\boldsymbol{E}_{\text{D}}, \boldsymbol{E}_{\text{S}}, \boldsymbol{R}_{\text{S}}, \boldsymbol{P}^e\}$ is a competitive equilibrium for the \$-market, $\{\boldsymbol{E}_{\text{D}}, \boldsymbol{E}_{\text{S}}\}$ maximizes the social planner's problem for the \$-market. By the KKT conditions we obtain,

$$\begin{aligned} 0 = \frac{\partial \mathcal{L}^h}{\partial E_{\text{D}n}} = & -\nabla v_n(E_{\text{D}n}) - \nabla c^{\text{bo}}_n(D_n - E_{\text{D}n}) \\ & + \lambda + \sum_l (\mu^-_{ln} - \mu^+_{ln}) \cdot H_{ln} + \eta^+_n - \eta^-_n \end{aligned}$$

$$0 = \frac{\partial \mathcal{L}^h}{\partial E_{sn}} = P^e_n - \lambda - \sum_l (\mu^-_{ln} - \mu^+_{ln}) \cdot H_{ln}$$

On summing these two equations we obtain,

$$P^e_n = \nabla v_n(E_{\text{D}n}) + \nabla c^{\text{bo}}_n(D_n - E_{\text{D}n}) - \eta^+_n + \eta^-_n$$

The proposition then follows from the complementary slackness conditions. □

It appears that prices depend upon the actions of the players in (35). This is not the case: Just as in Prop. 11 we can write the price as $P^e_n(t) = \nabla v_n(E^*_{\text{D}n}(t)) + \nabla c^{\text{bo}}_n(D_n(t) - E^*_{\text{D}n}(t)) + \Lambda^*(t)$, where $\{E^*_{\text{D}n}(t)\}$ (and consequently $\Lambda^*(t)$) are obtained as the solution to the SPP.

To apply the proposition we must identify the parameters $\{\eta^-_n, \eta^+_n\}$. This is possible since they are precisely sensitivities of the social planner's problem with respect to the respective constraints $E_{\text{D}n} \ge 0$ and $E_{\text{D}n} \le D_n$. In general the equations given in Prop. 18 may not be sufficient to obtain a closed form expression for the equilibrium prices. However, in the next set of examples we find that calculation is possible using the proposition.

Example: Three bus network Consider the network shown in Fig. 1. We will see that network constraints may cause prices to rise *beyond* what can be found in the single-bus model, and that prices can be negative.

We assume that the impedances are identical in the three transmission lines, and that bus 1 is the reference bus, giving the expression (12) for H. Each node has linear utility of consumption and disutility of blackout (11), with common parameters v, c^{bo}.

The snapshot social planner's problem for the \$-market is

$$\begin{aligned} \min \quad & -\big[v(E_{\text{D}2} + E_{\text{D}3}) - c^{\text{bo}}(200 - E_{\text{D}2} - E_{\text{D}3}) \big] \\ \text{s.t.} \quad & E_{\text{S}1} = E_{\text{D}2} + E_{\text{D}3} \\ & -f^{\max}_{12} \le \tfrac{2}{3} E_{\text{D}2} + \tfrac{1}{3} E_{\text{D}3} \le f^{\max}_{12} \\ & -f^{\max}_{23} \le \tfrac{1}{3} E_{\text{D}2} - \tfrac{1}{3} E_{\text{D}3} \le f^{\max}_{23} \\ & -f^{\max}_{13} \le \tfrac{1}{3} E_{\text{D}2} + \tfrac{2}{3} E_{\text{D}3} \le f^{\max}_{13} \\ & 0 \le E_{\text{D}2} \le 170\,, \ \ 0 \le E_{\text{D}3} \le 30 \end{aligned}$$

In the two special cases that follow we fix a specific time t, and suppose that $D_2 = 170$ MW, $D_3 = 30$ MW. We suppose moreover that the supporting price P^e_1 at bus 1 is *zero*, and that this is true not just for the snapshot values $\{E_{\text{D}i}, E_{\text{S}i}, R_{\text{S}i}\}$ and parameters $\{f^{\max}_{ij}\}$, but for all values in a neighborhood of these nominal values. This is not unreasonable given the results of Sec. 4.1, provided that reserves are strictly positive at bus 1.

Under these assumptions we can then compute the prices at the other two buses.

Negative prices Assume that $f^{\max}_{23} = 40$ MW, while the other two lines are unconstrained.

Solving the social planner's problem for the \$-market we obtain $E_{\text{D}2} = 150$ MW, $E_{\text{D}3} = 30$ MW, and we find that the constraint $f^{\max}_{23} = 40$ MW is reached. Since $0 < E_{\text{D}2} < D_2$, we have $P^e_2 = v + c^{\text{bo}}$.

For a given $\epsilon \in \mathbb{R}$ we perturb the constraint on $E_{\text{D}3}$ to obtain $0 \le E_{\text{D}3} \le 30 + \epsilon$. On re-solving the SPP we obtain $E_{\text{D}2} = 150 + \epsilon$ MW, and $E_{\text{D}3} = 30 + \epsilon$ MW. Applying Prop. 18 we conclude that P^e_1 is given by the limit,

$$v + c^{\text{bo}} + \lim_{\epsilon \to 0} \frac{-(180 + 2\epsilon)v + (20 - 2\epsilon)c^{\text{bo}} + 180v - 20c^{\text{bo}}}{\epsilon}$$

which is the *negative* value $P^e_1 = -(v + c^{\text{bo}})$.

Prices exceeding the choke up price. Assume that $f^{\max}_{13} = 50$ MW, while the other two lines are unconstrained.

Solving the social planner's problem for the $\mathcal{B}$-market we obtain $E_{\mathrm{D}2} = 150$ MW, $E_{\mathrm{D}3} = 0$ MW, and the constraint $f_{13}^{\max} = 50$ MW is reached. Prop. 18 gives $P_2^e = v + c^{\mathrm{bo}}$ since $0 < E_{\mathrm{D}2} < D_2$.

For a given $\epsilon \in \mathbb{R}$ we perturb the constraint on $E_{\mathrm{D}3}$ to obtain $0+\epsilon \le E_{\mathrm{D}3} \le 30$. On re-solving the SPP we obtain $E_{\mathrm{D}2} = 150 - 2\epsilon$ MW $E_{\mathrm{D}3} = \epsilon$ MW. We conclude that P_2^e is again expressed as a limit,

$$v + c^{\mathrm{bo}} + \lim_{\epsilon \to 0} \frac{-(180-\epsilon)v + (20+\epsilon)c^{\mathrm{bo}} + 180v - 20c^{\mathrm{bo}}}{\epsilon}$$

That is, $P_2^e = 2(v + c^{\mathrm{bo}})$.

5. CONCLUSIONS

The control systems field is a rich source of insights, analysis, and algorithms to be deployed in electricity market design. We introduced in this paper a *dynamic* competitive equilibria model for power systems based on a theoretical scaffolding familiar to the decision and control systems community. We arrive at long-recognized economic conclusions, such as the characterization of equilibrium prices in terms of marginal cost. We strongly diverge from the economics field since these result hold only on average. High volatility of prices can have negative impacts on consumers, suppliers or both. This is most obvious from the supplier's point of view: If my average prices will only meet my average marginal cost, and my income will be so uncertain, why would I make the enormous investments required to go into this business?

This brings us to a significant asymmetry between the notion of equilibrium in economics and in control: The equilibrium point is frequently taken as an *end point* in an economic analysis. It is the ultimate optimal point to which we all wish to converge. In any course on control, an equilibrium is only a *starting point*. After determining a desirable equilibrium we next consider its stability, a region of asymptotic stability, and we attempt to estimate robustness to un-modeled dynamics. We then simulate, and if our results do not match our predictions we redesign the control algorithm, or the system we are attempting to control. Can we utilize some of the insights and ideas from control in the design of electricity markets? We do believe the answer is yes. We hope that the ideas described in this work will motivate the control community and will form a building block for constructing and analyzing far more intricate models, taking into account a broader range of issues which will help to improved market design.

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